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POISSON SPECTRAL ESTIMATION USING HARDLIMITED SAMPLES.(U)

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POISSON SPECTRAL ESTIMATION USING HARDLIMITED SAMPLES

DM Klamer

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NAVAL OCEAN SYSTEMS CENTER
SAN DIEGO, CALIFORNIA 92152

NAVAL OCEAN SYSTEMS CENTER, SAN DIEGO, CA 92152

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In this report, the problem of reconstructing the spectral density of a Gaussian signal process from hard-limited observations taken at Poisson sampling instants was considered. The estimate was shown to be asymptotically unbiased as the number of observation points approached infinity. In addition, the covariance of the estimate was also asymptotically bounded as the number of observations tended to infinity. The focus of future work will be to obtain tighter bounds on the covariance of the estimate and to show that these bounds tend to zero as the number of observations tends to infinity.

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SUMMARY

In this report, the problem of reconstructing the power spectral density of a Gaussian signal process from hardlimited observations taken at Poisson sampling instants was considered. Hardlimiting the signal represents the most drastic form of amplitude quantization that is possible – but yet allows meaningful analysis to be performed. The sampled hardlimited data is represented with only one bit of information which gives a significant reduction in the amount of storage area needed for retaining the data. Also, the processing of the data becomes simplified because of the simple representation. By Theorem 1, the power spectral density estimate, $\hat{\phi}_N(\lambda)$, is shown to be asymptotically unbiased as the number of observation points approaches infinity. The asymptotic rate of convergence for the bias (of the estimate) is identical to the rates of convergence for the cases of Poisson spectral estimation (without hardlimiting) and of periodic spectral estimation (to the aliased spectral density). Furthermore, because the estimate is asymptotically unbiased, no aliasing of the spectral density occurs in spite of the hardlimiting.

The covariance of the estimate was shown, however, only to be bounded as the number of observations tends to infinity. Although not indicated by the analysis up to this point (primarily Theorem 1), the number of hardlimited samples needed for the same quality of estimate of the spectral density may increase significantly over the number of samples needed in the case when no hardlimiting is done. The focus of future work will be to obtain tighter bounds on the covariance of the estimate and to show that these bounds tend to zero as the number of observations tends to infinity.

CONTENTS

INTRODUCTION

This paper is concerned with the estimation of the spectral density function of a continuous-time stochastic process from observations taken at discrete instants of time after the process has been hardlimited. In [1], a continuous-time estimate using the hardlimited realization of a stationary, Gaussian stochastic process was shown to be a consistent estimate of the spectral density function of the original process. Hinich [2] deals with the consistent estimation of aliased versions of nonbandlimited spectra from hardlimited Gaussian processes. Brillinger [3] provides consistent estimates of the cross-spectrum and spectra of stationary bivariate Gaussian processes from the zeros of the processes. Finally, Masry and Lui [4] and Masry [5] provide consistent estimates of spectra of continuous-time processes from observations taken at Poisson sampling instants.

In this paper we consider the problem of estimating the spectral density function of a Gaussian process – the estimate is based upon hardlimited (+1, –1 or 0) samples of the process taken at Poisson sampling instants. The proposed estimate is shown to be asymptotically unbiased. For the covariance of the estimate, all but two terms are shown to go to zero and these terms are easily shown to be bounded. Future work includes finding bounds for the rate at which the bias asymptotically goes to zero, along with showing that the covariance goes to zero, and the rate at which the covariance tends to zero.

Let $X = \{X(t), -\infty < t < \infty\}$ be a real fourth-order, stationary Gaussian process with zero mean, correlation function $R(\tau)$ and spectral density function $\phi(\lambda)$. Define the hardlimited version of the process $X(t)$ as $Y(t) = \operatorname{sgn} X(t)$ where sgn X is equal to 1, –1 or 0 according to X being positive, negative or zero, respectively. Then Y(t) is a fourth-order, zero mean, wide-sense stationary process with correlation function [3,6]

$$\begin{aligned} R_Y(\tau) &= E[Y(t+\tau)\,Y(t)] \\ &= (2/\pi)\,\mathrm{Arcsin}\,(R(\tau)/R(0)). \end{aligned} \tag{1}$$

Since all amplitude information is lost when the X process is hardlimited we assume that $R(\tau)$ is normalized such that $R(0) = 1$.

The following relationships hold:

$$R(t) = \int_{-\infty}^{\infty} \phi(\lambda)\, e^{it\lambda}\, d\lambda$$

$$R_Y(t) = \int_{-\infty}^{\infty} \phi_Y(\lambda)\, e^{it\lambda}\, d\lambda \tag{2}$$

and if $R(t) \epsilon L_1$ and $R_Y(t) \epsilon L_1$ then the inverse relationships of (2) hold

$$\phi(\lambda) = \int_{-\infty}^{\infty} R(t)\, e^{-it\lambda}\, dt/2\pi$$

$$\phi_Y(\lambda) = \int_{-\infty}^{\infty} R_Y(t)\, e^{-it\lambda}\, dt/2\pi\,. \tag{3}$$

Using (1) and (3) and noting that $R_Y(t)$ is an even function, the spectral density function can be written as

$$\phi(\lambda) = \int_{0}^{\infty} \sin\,[\pi\, R_Y(t)/2]\;\cos t\lambda\; dt/\pi\,. \tag{4}$$

The problem that is considered here is to estimate $\phi(\lambda)$ by sampling the hardlimited process $Y(t)$ at discrete instants of time,

$$\left\{t_n\right\}_{n=1}^{N},$$

which are determined by a Poisson point process.

The proposed Poisson finite sample estimate of $\phi(\lambda)$, using N discrete observations of $Y(t)$ which are *taken from a Poisson point process*, is

$$\hat{\phi}_N(\lambda) = \frac{1}{\pi\beta} \sum_{n=1}^{N} w_N(t_n)\, \cos \lambda t_n\, \sin\left\{\frac{\pi}{2N} \sum_{k=1}^{N-n} Y(t_{k+n})\, Y(t_k)\right\}\,. \tag{5}$$

The $\left\{t_n\right\}_{n=0}^{\infty}$ are generated by a Poisson point process on $(0,\infty)$, i.e.,

$$t_o = 0$$

$$t_n = t_{n-1} + \alpha_n \qquad n = 1, 2, 3, \ldots \tag{6a}$$

where the $\{\alpha_n\}$ are independent, identically-distributed (positive) random variables with a common exponential distribution $F(x) = 1 - e^{-\beta x}$ and β is the average sampling rate. The process $T = \{t_n\}$ is assumed to be independent of the process $X(t)$. The probability density function of $t_{k+n} - t_k$ is independent of k and is given by

$$f_n(t) = \beta \frac{(\beta - t)^{n-1}}{(n-1)!}\, e^{-\beta t}\,, \quad t \geqslant 0\,. \tag{6b}$$

The covariance averaging kernel, $w_N(t)$, is given by

$$w_N(t) = h\left(\frac{t}{M_N}\right)$$

where

$$h(t) = \int_{-\infty}^{\infty} H(\lambda)\, e^{it\lambda}\, d\lambda$$

and $H(\lambda) \epsilon L_1$ is even and integrates to 1. The spectral window, $W_N(\lambda)$, is defined by

$$W_N(\lambda) = M_N\, H(M_N \lambda)$$

where $M_N < N$ is such that $M_N \to \infty$ and $M_N/N \to 0$ as $N \to \infty$.

The primary results of this paper are concerned with the asymptotic properties of the spectral estimate (5). Theorem 1 establishes that the estimate is asymptotically unbiased as the number of observations N tends to infinity. Furthermore, under mild conditions of integrability on the correlation function ($tR(t) \epsilon L_1$), the asympototic rate of convergence of the estimate to the true spectral density (of the original nonbandlimited) signal process is O(1/N), i.e., the bias error tends to zero at the rate of *at least* 1/N. As a consequence of Theorem 1, the estimate $\hat{\phi}_N(\lambda)$ eliminates "aliasing" of the spectral density (the folding of high frequency power to lower frequencies which occurs with periodic sampling) in defiance to the hardlimiting. We also note that the asymptotic rate of decrease of the bias is identical to the asymptotic rate of decrease of the bias for Poisson spectral estimation when no hardlimiting occurs, which is, in turn, identical to the rate of decrease of the bias of continuous time spectral estimation [5]. The covariance of the estimate is shown, however, only to be bounded as the number of observations tends asymptotically to infinity.The bounding of the convariance is done in a series of lemmata, Lemma 2 through Lemma 5. We show that the covariance is (asymptotically) bounded by

$$\mathrm{Cov}\left[\hat{\phi}_N(\lambda), \hat{\phi}_N(\omega)\right] = O\left[\left(\beta \int_0^{\infty} |R(u)|\, du\right)^2 + \int_0^{\infty} R^2(u)\, du\right.$$

$$+\frac{\beta(1+\beta)}{4\beta^2}\left[\max[1, b(0)]\left(\int_0^\infty b(u)\,du\right)^2\right.$$

$$\left.+\left(\int_0^\infty |R_Y(u)|\,du\right)^2\right\} + \phi(\lambda)\phi(\omega)\Bigg]$$

We note that since the covariance of the estimate – at least by the analysis contained in this report – does not tend to zero, but is only bounded, the question as to whether or not the estimate is consistent is still open.

THE JOINT DENSITY FUNCTION FOR POISSON SAMPLES

Before stating the main results, we collect together in this section some preliminary results which will be helpful in the derivation of the bias and variance of the estimate (5). We begin by defining $\widehat{R}_Y^{(N)}(n)$ as

$$\widehat{R}_Y^{(N)}(n) = (1/N) \sum_{k=1}^{N-n} Y(t_{k+n})\,Y(t_k) \tag{7}$$

and the estimate (5) becomes

$$\widehat{\phi}_N(\lambda) = \frac{1}{\pi\beta} \sum_{n=1}^{N} w_N(t_n) \cos \lambda t_n \sin\left[\pi \widehat{R}_Y^{(N)}(n)/2\right]. \tag{8}$$

Note that $\widehat{R}_Y^{(N)}(n)$ is not an estimate of $R_Y(n)$ since

$$E\,\widehat{R}_Y^{(N)}(n) = \left(1 - \frac{n}{N}\right) \int_0^\infty R_Y(t)\,f_n(t)\,dt$$

and as $N \to \infty$ becomes

$$\int_0^\infty R_Y(t)\, f_n(t)\, dt ,$$

which in general is not equal to $R_Y(n)$ but can be considered as an estimate of $R_Y(t_n)$.

The following identities are easily established (recalling that $R(0) = 1$) from (1)

$$\sin\,[\pi R_Y(t)/2] = R(t)$$

$$\cos\,[\pi R_Y(t)/2] = [1 - R^2(t)]^{1/2} . \tag{9}$$

These identities are used later and also here to simplify the two term Taylor series expansion of $\sin\,[\pi \hat{R}_Y^{(N)}(n)/2]$ about the point $\pi R_Y(t_n)/2$

$$\sin\,[\pi \hat{R}_Y^{(N)}(n)/2]$$

$$= R(t_n) + A_n \pi \left[\hat{R}_Y^{(N)}(n) - R_Y(t_n)\right] \left[1 - R^2(t_n)\right]^{1/2}/2 \tag{10}$$

where $|A_n| \leqslant 1$. The Taylor series expansion is used to determine the asymptotic behavior of the covariance of the spectral density estimate.

Next we borrow a lemma from [4].

Lemma 1: Consider the integrals

$$d_n = \int_0^\infty g(x)\, f_n(x)\, dx , \qquad n = 1, 2, \ldots$$

where $g \in L_1$ and f_n is given by (6b). Then

(a) $$\sum_{n=1}^\infty d_n = \beta \int_0^\infty g(x)\, dx < \infty$$

and, if in addition $x\, g(x) \in L_1$, then

(b) $$\sum_{n=1}^\infty n\, d_n = \beta \int_0^\infty (1 + \beta x)\, g(x)\, dx < \infty .$$

The proof is straightforward and given in [4].

The result of the following proposition, the joint density function of the sampling points, is used in the derivation of the covariance of the estimates (5) and is independently of interest.

Proposition. Let $\{t_n\}$ be defined by (6a). Then the joint density function of t_n and t_{n+k}, $f_{n,n+k}(x,y)$, is given by

$$f_{n,n+k}(x,y) = \begin{cases} f_k(y-x)\, f_n(x) & y > x \\ 0 & y \leqslant x \end{cases}$$

for $n = 1, 2, \ldots,$ and $k = 1, 2, \ldots,$ where $f_n(x)$ is the density function (6b) of t_n.

Proof. The following two relationships are used in the derivation [6]

$$f_{n,n+k}(x,y) = f_{n+k}(y \mid t_n = x)\, f_n(x) \tag{11a}$$

$$f_{n+k}(y \mid t_n = x) = \frac{\partial}{\partial y} F_{n+k}(y \mid t_n = x) \tag{11b}$$

where $F_{n+k}(y \mid t_n = x)$ is the conditional distribution function of t_{n+k} given that $t_n = x$. From equation (6a), t_{n+k} is given by

$$t_{n+k} = t_n + A_{n,k}$$

where

$$A_{n,k} = \sum_{i=n}^{n+k-1} \alpha_i \ .$$

Now, by recalling the fact that $\{\alpha_i\}$ are independently, identically distributed

$$\begin{aligned} F_{n+k}(y \mid t_n = x) &= P(t_n + A_{n,k} \leqslant y \mid t_n = x) \\ &= P(A_{n,k} \leqslant y - x) \\ &= F_k(y - x) \ . \end{aligned}$$

Using (11b), $f_{n+k}(y \mid t_n = x) = f_k(y-x)$ and substituting this into (11a) gives the desired result.

Q.E.D.

BIAS OF THE ESTIMATOR

In this section we show that the bias of the estimator goes to zero asymptotically as $N \to \infty$. The bias of the estimator (5) is defined as

$$b\left[\widehat{\phi}_N(\lambda)\right] = E\left[\widehat{\phi}_N(\lambda)\right] - \phi(\lambda) .$$

The following theorem establishes that the estimator is asymptotically unbiased.

Theorem 1. Let $R(t) \epsilon L_1$. The estimate $\widehat{\phi}_N(\lambda)$ is an asymptotically unbiased estimate of $\phi(\lambda)$, i.e.,

$$\text{(a)} \quad b\left[\widehat{\phi}_N(\lambda)\right] = o(1)$$

uniformly in λ as $N \to \infty$. Furthermore, if $t\, R(t) \epsilon L_1$, then

$$\text{(b)} \quad b\left[\widehat{\phi}_N(\lambda)\right] = O(1/N) ,$$

uniformly in λ as $N \to \infty$.

Proof. By (10) the estimator becomes equal to

$$\widehat{\phi}_N(\lambda) = Q_1(\lambda,N) + Q_2(\lambda,N) \tag{12}$$

where

$$Q_1(\lambda,N) = \frac{1}{\beta\pi} \sum_{n=1}^{N} w_N(t_n) \cos \lambda t_n\, R(t_n)$$

$$Q_2(\lambda,N) = \frac{1}{2\beta} \sum_{n=1}^{N} A_n w_N(t_n) \cos \lambda\, t_n \left[\widehat{R}_Y^{(N)}(n) - R_Y(t_n)\right] \left[1 - R^2(t_n)\right]^{1/2} .$$

Now taking the expected value of $Q_1(\lambda,N)$ (with respect to $T = \{t_n\}$ since the Y process does not appear)

$$E\, Q_1(\lambda,N) = \frac{1}{\beta\pi} \int_0^{\infty} w_N(t) \cos \lambda t\, R(t) \sum_{n=1}^{N} f_n(t)\, dt \tag{13}$$

and

$$\lim_{N\to\infty} w_N(t) \sum_{n=1}^{N} f_n(t) = \beta$$

since $\lim w_N(t) = h(o) = 1$ and by Lemma 1 $\lim \Sigma f_n(t) = \beta$. Also note that

$$|w_N(t) \cos \lambda t \sum_{n=1}^{N} f_n(t)| \leqslant \beta \cdot H_a$$

where $H_a = \int |H(\lambda)|\, d\lambda$. Therefore, by the Lebesgue dominated convergence theorem

$$\lim_{N\to\infty} E\, Q_1(\lambda,N) = \frac{1}{\beta\pi} \int_o^\infty \lim_{N\to\infty} w_N(t) \sum_{n=1}^{N} f_n(t) \cos \lambda t\, R(t)\, dt$$

$$= \frac{1}{\pi} \int_o^\infty \cos \lambda t\, R(t)\, dt = \phi(\lambda)\ . \tag{14}$$

To take the expected value of $Q_2(\lambda,N)$, the expectation is first taken with respect to the Y process and then with respect to the T process. Taking the expected value with respect to Y yields the bound

$$|E_Y Q_2(\lambda,N)| \leqslant H_a \sum_{n=1}^{N} \left| \left[\frac{1}{N} \sum_{k=1}^{N-n} R_Y(t_{k+n} - t_k)\right] - R_Y(t_n) \right| \tag{15}$$

since $|w_N(t)| \leqslant H_a$ and $[1 - R^2(t)]^{1/2} \leqslant 1$. Now taking the expected value of (15) with respect to T gives

$$|E\, Q_2(\lambda,N)| \leqslant H_a \left| \sum_{n=1}^{N} \left\{ \left(1 - \frac{n}{N}\right) \int_o^\infty R_Y(t)\, f_n(t)\, dt - \int_o^\infty R_Y(t)\, f_n(t)\, dt \right\} \right|$$

$$= \frac{H_a}{N} \sum_{n=1}^{N} n \int_o^\infty R_Y(t)\, f_n(t)\, dt \tag{16}$$

and by the Kronecker lemma [6], since

$$\lim \sum_{n=1}^{N} \int_{o}^{\infty} R_Y(t)\, f_n(t)\, dt = \beta \int_{o}^{\infty} R_Y(t)\, dt < \infty ,$$

the expected value of the $Q_2(\lambda,N)$ term goes to zero:

$$|E\, Q_2(\lambda,N)| \to 0 \text{ as } N \to \infty \ . \tag{17}$$

Therefore, the estimator (5) is asymptotically unbiased and part (a) is established.

To establish part (b), the bias can be written, using (12) through (14), as

$$b\left[\hat{\phi}_N(\lambda)\right] = b_1(\lambda,N) + E\, Q_2(\lambda, N)$$

where $b_1(\lambda,N)$ is defined by

$$b_1(\lambda,N) = \frac{-1}{\pi\beta} \int_{o}^{\infty} w_N(t) \cos \lambda t\, R(t) \sum_{n=N+1}^{\infty} f_n(t)\, dt \ .$$

Let

$$e_n = \int_{o}^{\infty} |R(t)|\, f_n(t)\, dt \qquad n = 1, 2, \ldots$$

then $\left\{n\, e_n\right\}$ is summable by Lemma 1 (b). Hence,

$$|b_1(\lambda,N)| \leq \frac{1}{\pi\beta} \sum_{n=N}^{\infty} e_n \leq \frac{1}{\pi\beta N} \sum_{n=N}^{\infty} n\, e_n = \frac{1}{N}\, o(1)$$

and by (16)

$$|E\, Q_2(\lambda,N)| = O(1/N) \ .$$

Therefore,

$$b\left[\hat{\phi}_N(\lambda)\right] = O(1/N)$$

and part (b) is established. Q.E.D.

COVARIANCE OF THE ESTIMATOR

The covariance of the estimator (5) is given by

$$\mathrm{Cov}\left[\hat{\phi}_N(\lambda), \hat{\phi}_N(\omega)\right] = E\left\{\left[\hat{\phi}_N(\lambda) - E\hat{\phi}_N(\lambda)\right]\left[\hat{\phi}_N(\omega) - E\hat{\phi}_N(\omega)\right]\right\}.$$

By replacing $E\hat{\phi}_N(\lambda)$ with the bias expression, along with noting that the bias is not a random variable, the covariance becomes

$$\mathrm{Cov}\left[\hat{\phi}_N(\lambda), \hat{\phi}_N(\omega)\right] = E\left[\hat{\phi}_N(\lambda) - \phi(\lambda)\right]\left[\hat{\phi}_N(\omega) - \phi(\omega)\right] - b\left[\hat{\phi}_N(\lambda)\right] b\left[\hat{\phi}_N(\omega)\right].$$

Since the bias goes to zero asymptotically as $N \to \infty$ we need only to investigate the asymptotic behavior of $E[\hat{\phi}_N(\lambda) - \phi(\lambda)]\,[\hat{\phi}_N(\omega) - \phi(\omega)]$. By using the two term Taylor series expansion (10) of $\sin[\pi \hat{R}_Y^{(N)}(n)/2]$ we have

$$E\left[\hat{\phi}_N(\lambda) - \phi(\lambda)\right]\left[\hat{\phi}_N(\omega) - \phi(\omega)\right] = E\left\{\frac{1}{\beta\pi}\sum_{n=1}^{N} w_N(t_n)\cos\lambda t_n R(t_n)\right.$$

$$\left. + \frac{1}{2\beta}\sum_{n=1}^{N} A_N w_N(t_n)\cos\lambda t_n\left[\hat{R}_Y^{(N)}(n) - R_Y(t_n)\right]\left[1 - R^2(t_n)\right]^{1/2} - \phi(\lambda)\right\}$$

$$\cdot\left\{\frac{1}{\beta\pi}\sum_{n=1}^{N} w_N(t_n)\cos\omega t_n R(t_n)\right.$$

$$\left. + \frac{1}{2\beta}\sum_{n=1}^{N} B_N w_N(t_n)\cos\omega t_n\left[\hat{R}_Y^{(N)}(n) - R_Y(t_n)\right]\left[1 - R^2(t_n)\right]^{1/2} - \phi(\omega)\right\}$$

$$= \sum_{i=1}^{4} T_i(N, \lambda, \omega)$$

where

$$T_1(N,\lambda,\omega) = E\left[\frac{1}{\beta\pi}\sum_{n=1}^{N} w_N(t_n)\cos\lambda t_n\, R(t_n) - \phi(\lambda)\right]$$

$$\cdot\left[\frac{1}{\beta\pi}\sum_{n=1}^{N} w_N(t_n)\cos\omega t_n\, R(t_n) - \phi(\omega)\right]$$

$$T_2(N,\lambda,\omega) = E\left\{\frac{1}{2\beta}\sum_{n=1}^{N} A_N w_N(t_n)\cos\lambda t_n\left[\widehat{R}_Y^{(N)}(n) - R_Y(t_n)\right]\left[1-R^2(t_n)\right]^{1/2}\right\}$$

$$\cdot\left\{\frac{1}{\beta\pi}\sum_{n=1}^{N} w_N(t_n)\cos\omega t_n\, R(t_n) - \phi(\omega)\right\}$$

$$T_3(N,\lambda,\omega) = E\left\{\frac{1}{2\beta}\sum_{n=1}^{N} B_N w_N(t_n)\cos\omega t_n\left[\widehat{R}_Y^{(N)}(n)-R_Y(t_n)\right]\left[1-R^2(t_n)\right]^{1/2}\right\}$$

$$\cdot\left\{\frac{1}{\beta\pi}\sum_{n=1}^{N} w_N(t_n)\cos\lambda t_n\, R(t_n) - \phi(\lambda)\right\}$$

$$T_4(N,\lambda,\omega) = E\left\{\frac{1}{2\beta}\sum_{n=1}^{N} A_N w_N(t_n)\cos\lambda t_n\left[\widehat{R}_Y^{(N)}(n)-R_Y(t_n)\right]\left[1-R^2(t_n)\right]^{1/2}\right\}$$

$$\cdot\left\{\frac{1}{2\beta}\sum_{n=1}^{N} B_N\, w_N(t_n)\cos\omega t_n\left[\widehat{R}_Y^{(N)}(n) - R_Y(t_n)\right]\left[1-R^2(t_n)\right]^{1/2}\right\}.$$

The asymptotic behavior of each $T_i(N,\lambda,\omega)$ will be obtained in a series of lemmata. We begin with the $T_2(N,\lambda,\omega)$ and $T_3(N,\lambda,\omega)$ terms.

Lemma 2. Let $R(t)\in L_1$ and $R_Y(t)\in L_1$. Then $T_2(N,\lambda,\omega)$ goes to zero asymptotically as N goes to infinity.
Proof. The second term of the product in $T_2(N,\lambda,\omega)$ is (recall (12)) equal to

$$\phi(\omega)\, Q_2(\lambda,N)$$

and by (17) goes to zero asymptotically as $N \to \infty$. The first term of the product is, after taking the expectation, equal to

$$\frac{A_N}{2\beta^2\pi} \int_0^\infty \int_0^\infty w_N(s)\, w_N(t) \cos \lambda s \cos \omega t \; R_Y(s)\, [1 - R^2(s)]\, R(t)$$

$$\cdot \sum_{n=1}^{N} \sum_{k=1}^{N} \frac{n}{N} f_{n,k}(s,t)\, ds\, dt$$

where $f_{n,k}(s,t)$ is given by the Proposition. This term is bounded by

$$\frac{1}{2\beta^2\pi} \int_0^\infty \int_0^\infty R_Y(s)\, R(t)\, \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{N} n\, f_{n,k}(s,t)\, ds\, dt\ .$$

Consider the sum

$$S(N,s,t) = \sum_{n=1}^{N} \sum_{k=1}^{N} f_{n,k}(s,t)$$

$$= \sum_{n=1}^{N-1} \sum_{k=1}^{N-n} [f_{n,n+k}(s,t) + f_{n,n+k}(t,s)] + \sum_{n=1}^{N} f_n(\min\{s,t\})$$

for the case $s \leqslant t$. Then

$$S(N,s,t) = \sum_{n=1}^{N-1} \sum_{k=1}^{N-n} f_k(t-s)\, f_n(s) + \sum_{n=1}^{N} f_n(s)$$

$$\leqslant \sum_{n=1}^{N-1} f_n(s) \sum_{k=1}^{N-1} f_k(t-s) + \sum_{n=1}^{N} f_n(s)$$

which converges to $\beta(1+\beta)$ as $N \to \infty$ and by the Kronecker lemma [6]

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{k=1}^{N} n\, f_{n,k}(s,t) \to 0 \text{ as } N \to \infty .$$

The same argument holds for $t \leqslant s$ and, therefore, by the Lebesgue dominated convergence theorem

$$\int_0^\infty \int_0^\infty R_Y(s)\, R(t) \frac{1}{N} \sum_{n=1}^{N}\sum_{k=1}^{N} n\, f_{n,k}(s,t)\, ds\, dt \to 0$$

as $N \to \infty$. Hence

$$T_2(N,\lambda,\omega) \to 0 \text{ as } N \to \infty . \qquad \text{Q.E.D.}$$

Lemma 3. Let $R(t) \in L_1$ and $R_Y(t) \in L_1$. Then

$$T_3(N,\lambda,\omega) \to 0$$

as $N \to \infty$.

The proof of Lemma 3 is identical to the proof of Lemma 2.

By multiplying the factors of $T_1(N,\lambda,\omega)$ we obtain

$$T_1(N,\lambda,\omega) = \sum_{i=1}^{4} T_{1,i}(N,\lambda,\omega)$$

where

$$T_{1,1}(N,\lambda,\omega) = \frac{1}{(\beta\pi)^2} E\left\{ \sum_{n=1}^{N} w_N(t_n) \cos \lambda t_n\, R(t_n) \cdot \sum_{n=1}^{N} w_N(t_n) \cos \omega t_n\, R(t_n) \right\}$$

$$T_{1,2}(N,\lambda,\omega) = \frac{-\phi(\lambda)}{\beta\pi} E\left\{ \sum_{n=1}^{N} w_N(t_n) \cos \omega t_n\, R(t_n) \right\}$$

$$T_{1,3}(N,\lambda,\omega) = T_{1,2}(N,\omega,\lambda)$$

$$T_{1,4}(N,\lambda,\omega) = \phi(\lambda)\, \phi(\omega) .$$

We note that $T_{1,2}(N,\lambda,\omega) = -\phi(\lambda) \cdot Q_1(\omega,N)$ and by the proof of Theorem 1

$$\lim_{N\to\infty} T_{1,2}(N,\lambda,\omega) = -\phi(\lambda)\,\phi(\omega) \tag{18}$$

The term $T_{1,1}(N,\lambda,\omega)$ is divided into three sums:

$$T_{1,1}(N,\lambda,\omega) = \sum_{i=1}^{3} T_{1,1,i}(N,\lambda,\omega)$$

where

$$T_{1,1,1}(N,\lambda,\omega) = \frac{1}{(\beta\pi)^2}\, E \sum_{n=1}^{N} w_N^2(t_n) \cos \lambda t_n \cos \omega t_n\, R^2(t_n)$$

$$T_{1,1,2}(N,\lambda,\omega) = \frac{1}{(\beta\pi)^2}\, E \sum_{n=1}^{N} \sum_{k>n} w_N(t_n)\, w_N(t_k) \cos \lambda t_n \cos \omega t_k\, R(t_n)\, R(t_k)$$

$$T_{1,1,3}(N,\lambda,\omega) = \frac{1}{(\beta\pi)^2}\, E \sum_{n=1}^{N} \sum_{k<n} w_N(t_n)\, w_N(t_k) \cos \lambda t_n \cos \omega t_k\, R(t_n)\, R(t_k) \,.$$

The $T_{1,1,1}$ term will be disposed of first. By taking the expected value (with respect only to the $\{t_k\}$)

$$T_{1,1,1}(N,\lambda,\omega) = \frac{1}{(\beta\pi)^2} \sum_{k=1}^{N} \int_0^\infty w_N^2(t) \cos \lambda t \cos \omega t\, R^2(t)\, f_k(t)\, dt$$

and by the Lebesgue dominated convergence theorem

$$\lim_{N\to\infty} T_{1,1,1}(N,\lambda,\omega) = \frac{1}{\beta\pi^2} \int_0^\infty \cos \lambda t \cos \omega t\, R^2(t)\, dt \,. \tag{19}$$

We note that $T_{1,1,2}$ and $T_{1,1,3}$ can be rewritten as

$$T_{1,1,2}(N,\lambda,\omega) = \frac{1}{(\beta\pi)^2}\, E \sum_{n=1}^{N-1} \sum_{s=1}^{N-n} w_N(t_n)\, w_N(t_{n+s}) \cos\lambda t_n \cos\omega t_{n+s}\, R(t_n) R(t_{n+s})$$

$$= \frac{1}{(\beta\pi)^2} \int_o^\infty \int_o^\infty w_N(u)\, w_N(v) \cos\lambda u \cos\omega v\, R(u)\, R(v)$$

$$\cdot \sum_{n=1}^{N-1} \sum_{s=1}^{N-n} f_{n,n+s}(u,v)\, du\, dv$$

and

$$T_{1,1,3}(N,\lambda,\omega) = \frac{1}{(\beta\pi)^2} \int_o^\infty \int_o^\infty w_N(u)\, w_N(v) \cos\lambda u \cos\omega v\, R(u)\, R(v)$$

$$\cdot \sum_{n=1}^{N-1} \sum_{s=1}^{N-n} f_{n+s,n}(u,v)\, du\, dv \quad .$$

Now (with a change of variables in $T_{1,1,1}$ of u=v and v=u)

$$| T_{1,1,2}(N,\lambda,\omega) + T_{1,1,3}(N,\lambda,\omega) |$$

$$= \left| \int_o^\infty \int_u^\infty w_N(u)\, w_N(v)\, R(u)\, R(v) (\cos\lambda u \cos\omega v + \cos\lambda v \cos\omega u) \right.$$

$$\left. \cdot \sum_{n=1}^{N-1} \sum_{s=1}^{N-n} f_s(v-u)\, f_n(u)\, dv\, du \right|$$

$$\leqslant \int_o^\infty | R(u) | \sum_{n-1}^{N-1} f_n(u) \int_u^\infty | R(v) | \sum_{s=1}^{N-1} f_n(v-u)\, dv\, du$$

$$\leqslant \left(\beta \int_o^\infty | R(u) |\, du \right)^2 . \qquad (20)$$

We, therefore, have the following lemma:

Lemma 4. Let $R(t) \in L_1 \cap L_2$. Then

$$|T_1(N,\lambda,\omega)| \leq \left(\beta \int_0^\infty |R(u)|\, du\right)^2 + \int_0^\infty R^2(u)\, du + \phi(\lambda)\,\phi(\omega).$$

The proof of Lemma 4 follows (18) through (20).

The final term that we investigate is $T_4(N,\lambda,\omega)$. Before stating the derivation of the asymptotic behavoir of T_4, the following assumption on the fourth order moment of the output, hardlimited signal is made.

Assumption 1: The fourth order moment function of the output signal,

$$M_Y(t_1,t_2,t_3) \stackrel{\Delta}{=} E\left\{Y(t)\,Y(t+t_1)\,Y(t+t_2)\,Y(t+t_3)\right\},$$

satisfies

$$|M_Y(t_1,t_2,t_3)| \leq \prod_{i=1}^{3} b_i(t_i) \leq \prod_{i=1}^{3} b(t_i)$$

where $b_i(t) \in L_1$ and $b(t) \in L_1$ and b_i,b are continuous, even, nonnegative functions which are nonincreasing over $[0,\infty)$. In addition, the function $b(t)$ satisfies the inequality

$$b(t_1)\cdot b(t_2) \leq b(t_1 + t_2) \qquad (21)$$

for t_1 and t_2 positive.

We first note that the t_i's in the above assumption are real numbers and not the random variables from the Poisson point process T. Secondly, the inequality (and other conditions) for the function $b(t)$ in (21) is satisfied for the exponential function $\exp(-a|t|)$.

Lemma 5. Let $R_Y(t) \in L_1$. Then

$$|T_4(N,\lambda,\omega)| \leq \frac{\beta(1+\beta)}{4\beta^2}\left\{\max\left[1, b(0)\right]\left(\int_0^\infty b(u)du\right)^2 + \left(\int_0^\infty |R_Y(u)|\,du\right)^2\right\} + O(1).$$

The proof of Lemma 5 comprises Appendix A.

We note that by the above analysis, i.e., by combining the results of Lemmata 2 through 5, the estimate (5) is not a consistent estimate of the true spectral density $\phi(\lambda)$. In fact, the covariance of the estimate is (asymptotically) bounded by

$$\mathrm{Cov}\left[\widehat{\phi}_N(\lambda), \widehat{\phi}_N(\omega)\right] = O\left[\left(\beta \int_0^\infty |R(u)|\,du\right)^2 + \int_0^\infty R^2(u)\,du\right.$$

$$+ \frac{\beta(1+\beta)}{4\beta^2}\left\{\max\,[1, b(0)]\left(\int_0^\infty b(u)\,du\right)^2\right.$$

$$\left.\left. + \left(\int_0^\infty |R_Y(u)|\,du\right)^2\right\} + \phi(\lambda)\phi(\omega)\right].$$

The objective of future work is to obtain tighter bounds and show that these new bounds go to zero asymptotically as N tends to infinity. Specifically, tighter asymptotic bounds need to be found for the two terms $T_1(N, \lambda, \omega)$ and $T_{4,1}(N, \lambda, \omega)$.

CONCLUSIONS

The problem of reconstructing the power spectral density of a Gaussian signal process from hardlimited observations taken at Poisson sampling instants was considered. The estimate of the power spectral density of the original (not hardlimited) signal process is shown to be asymptotically unbiased as the number of observation points approaches infinity. The important consequences of the asymptotically unbiased estimator are as follows.

1. No aliasing of the spectral density of the (original) signal process occurs in spite of hardlimiting the amplitude of the signal.

2. The asymptotic rate of convergence of the bias is identical to the rates of convergence for the cases of (regular) Poisson spectral estimation and of periodic spectral estimation (that is, the convergence of the periodic estimate to the aliased spectral density). In particular, under mild integrability conditions on the power spectral density, the rate of convergence, for the bias, will be $O(1/N)$ where N is the number of observations.

The covariance of the estimate was shown, however, only to be bounded as the number of observations tends to infinity. Therefore, the crucial task that remains to be completed is to show that variance of the estimate tends asymptotically to zero. Upon the completion of bounding the variance of the estimate by a factor which tends to zero as the number of observations tends to infinity, the estimate of the spectral density will then be known to be mean-square consistent – a fact that will increase the confidence of the estimate. This task, the bounding of the variance, will be the focus of future work.

In addition to completing the proof of showing that the estimate is mean-square consistent, future work should include simulation results, for finite sample size N, that would indicate the feasibility of implementing Poisson spectral estimation using hardlimited samples. Furthermore, since each hardlimited sample requires only one bit of information for the purpose of storage or transmission, a trade-off study between the decrease in the required representation (of one bit per sample) which decreases storage area and the (possible) increase in the number of samples for good spectral estimates should be conducted.

REFERENCES

1. D. R. McNeil, "Estimating the covariance and spectral density functions from a clipped stationary time series," J. Royal Stat. Soc. (B), 29 (1967).

2. M. Hinich, "Estimation of spectra after hard clipping of Gaussian processes," Technometrics, 9 (1967).

3. D. R. Brillinger, "Estimation of the cross-spectrum of a stationary bivariate Gaussian process from its zeros," J. Royal Stat. Soc. (B), 30 (1968).

4. E. Masry and M. C. Lui, "Discrete-time spectral estimation of continuous-parameter processes – a new consistent estimate," IEEE Trans. Inform. Thry, IT-22 (1976).

5. E. Masry, "Poisson sampling and spectral estimation of continuous-time processes," IEEE Trans. Inform. Thry, IT-24 (1978).

6. A. Papoulsis, *Probability, Random Variables, and Stochastic Processes,* McGraw-Hill, (1965).

7. M. Loeve, *Probability Theory,* Springer-Verlag, (1977).

APPENDIX A

Lemma 5 (restated here for the convenience of the reader) and its proof comprise the information in this appendix.

Lemma 5. Let $R_Y(t) \epsilon L_1$. Then

$$|T_4(N,\lambda,\omega)| \leqslant \frac{\beta(1+\beta)}{4\beta^2} \left\{ \max\left[1, b(0)\right] \left(\int_0^\infty b(u)\,du \right)^2 + \left(\int_0^\infty |R_Y(u)|du \right)^2 \right\} + 0(1).$$

Proof. To begin the analysis of $T_4\,(N,\lambda,\omega)$, the expected value with respect to the Y process will be taken first – only the relevant terms (with respect to Y) appear.

$$E_Y \left[R_Y^{(N)}(n) - R_Y(t_n) \right] \left[R_Y^{(N)}(k) - R_Y(t_k) \right]$$

$$= \frac{1}{N^2} \sum_{\ell=1}^{N-n} \sum_{j=1}^{N-k} M_Y(t_j - t_\ell, t_{\ell+n} - t_\ell, t_{j+k} - t_\ell)$$

$$- R_Y(t_n)/N \cdot \sum_{\ell=1}^{N-k} R_Y(t_{\ell+k} - t_\ell)$$

$$- R_Y(t_k)/N \cdot \sum_{\ell=1}^{N-n} R_Y(t_{\ell+n} - t_\ell)$$

$$+ R_Y(t_n)\, R_Y(t_k)$$

and $T_4(N,\lambda,\omega)$ becomes

$$T_4(N,\lambda,\omega) = \left[T_{4,1}(N,\lambda,\omega) + T_{4,2}(N,\lambda,\omega) \right] / 4\beta^2$$

where

$$T_{4,1}(N,\lambda,\omega) = E_T \sum_{n=1}^{N} \sum_{k=1}^{N} A_N B_N w_N(t_n)\, w_N(t_k) \cos \lambda t_n \cos \omega t_k$$

$$\cdot \left[1 - R^2(t_n)\right]^{1/2} \left[1 - R^2(t_k)\right]^{1/2}$$

$$\cdot \left[\frac{1}{N^2} \sum_{\ell=1}^{N-n} \sum_{j=1}^{N-k} M_Y(t_j - t_\ell,\, t_{\ell+n} - t_\ell,\, t_{j+k} - t_\ell) \right.$$

$$\left. - R_Y(t_n)/N \cdot \sum_{\ell=1}^{N-k} R_Y(t_{\ell+k} - t_\ell) \right]$$

$$T_{4,2}(N,\lambda,\omega) = E_T \sum_{n=1}^{N} \sum_{k=1}^{N} A_N B_N\, w_N(t_n)\, w_N(t_k) \cos \lambda\, t_n \cos \omega t_k$$

$$\left|1 - R^2(t_n)\right|^{1/2} \left|1 - R^2(t_k)\right|^{1/2}$$

$$\cdot \left[R_Y(t_n)\, R_Y(t_k) - R_Y(t_k)/N \cdot \sum_{\ell=1}^{N-n} R_Y(t_{k+n} - t_\ell) \right].$$

The term $T_{4,2}(N,\lambda,\omega)$ is now considered and becomes

$$T_{4,2}(N,\lambda,\omega) = \sum_{n=1}^{N} \sum_{k=1}^{N} A_N B_N \int_0^\infty \int_0^\infty w_N(u)\, w_N(v) \cos \lambda u \cos \omega v$$

$$[1 - R^2(u)]^{1/2}\, [1 - R^2(v)]^{1/2}$$

$$\cdot \left[R_Y(u)\, R_Y(v) - R_Y(v)/N \cdot \sum_{\ell=1}^{N-n} R_Y(u) \right] f_{n,k}(u,v)\, du\, dv$$

$$= \frac{A_N B_N}{N} \sum_{n=1}^{N} \sum_{k=1}^{N} \int_0^\infty \int_0^\infty w_N(u)\, w_N(v) \cos \lambda u \cos \omega v$$

$$\cdot [1 - R^2(u)]^{1/2}\, [1 - R^2(v)]^{1/2}\, R_Y(u)\, R_Y(v)\, n\, f_{n,k}(u,v)\, du\, dv\ .$$

We recall from the proof of Lemma 2 that

$$\sum_{n=1}^{N} \sum_{k=1}^{N} f_{n,k}(u,v)$$

converges as N tends to infinity, and hence, by the Kronecker lemma [6]

$$\frac{1}{N} \sum_{n=1}^{N} n \sum_{k=1}^{N} f_{n,k}(u,v) \to 0 \text{ as } N \to \infty$$

and by the Lebesgue dominated convergence theorem

$$|T_{4,2}(N,\lambda,\omega)| \leq \int_0^\infty \int_0^\infty R_Y(u)\, R_Y(v)\, \frac{1}{N} \sum_{n=1}^{N} n \sum_{k=1}^{N} f_{n,k}(u,v)\, du\, dv$$

$$\to 0 \text{ as } N \to \infty\ .$$

We now consider $T_{4,1}(N,\lambda,\omega)$. The sum over the indices ℓ and j is divided into three regions

$$R_1 = \left\{ j,\ell: \quad \ell \leq j \right\}$$

$$R_2 = \left\{ j,\ell: \quad j < \ell < j + k \right\}$$

$$R_3 = \left\{ j,\ell: \quad j + k \leq \ell \right\}\ .$$

In R_1, $t_{j+k} - t_\ell \geq t_{j+k} - t_j \geq 0$ and, therefore,

$$b(t_{j+k} - t_\ell) \leq b(t_{j+k} - t_j)\ . \tag{A-1a}$$

Also note, that in R_1

$$b(t_j - t_\ell) \leq b(0)\ . \tag{A-1b}$$

In the region R_2, by Assumption 1

$$b(t_\ell - t_j)\, b(t_{j+k} - t_\ell) \leq b(t_{j+k} - t_j)\ . \tag{A-2}$$

Finally, in the region R_3, $t_\ell - t_j \geq t_{j+k} - t_j > 0$, and the following inequalities hold

$$b(t_j - t_\ell) \leq b(t_{j+k} - t_j) \tag{A-3a}$$

$$b(t_{j+k} - t_\ell) \leq b(0) \ . \tag{A-3b}$$

By Assumption 1 and the inequalities (A-1) through (A-3) $T_{4,1}$ (N,λ,ω) becomes bounded by

$$|T_{4,1}(N,\lambda,\omega)| \leq \sum_{n=1}^{N} \sum_{k=1}^{N} \int_0^\infty \int_0^\infty \left| w_N(u)\, w_N(v) \right|$$

$$\left\{ \frac{1}{N^2} \left[\sum_{R_1}\sum b(0)\, b(u)\, b(v) + \sum_{R_2}\sum b(u)\, b(v) + \sum_{R_3}\sum b(0)\, b(u)\, b(v) \right] \right.$$

$$\left. + |R_Y(u)\, R_Y(v)| \left(1 - \frac{k}{N}\right) \right\} f_{n,k}(u,v)\, du\, dv \ . \tag{A-4}$$

At this point we assume a specific covariance averaging kernel is given – namely the Dirichlet spectral window, which has as the corresponding covariance averaging kernel

$$w_N(t) = h(t/M_N) = \begin{cases} 1 & |t| \leq M_N \\ 0 & \text{otherwise} \ . \end{cases}$$

Thus, (A-4) is bounded by

$$|T_{4,1}(N,\lambda,\omega)| \leq \sum_{n=1}^{M_N} \sum_{k=1}^{M_N} \int_0^\infty \int_0^\infty$$

$$\left\{ \frac{1}{N^2} \left[\sum_{\ell=1}^{N-n} \sum_{j=1}^{N-k} 1 \right] \max\,[1, b(0)]\; b(u)\, b(v) \right.$$

$$\left. + |R_Y(u)\, R_Y(v)| \left(1 - \frac{k}{N}\right) \right\} f_{n,k}(u,v)\, du\, dv$$

$$= \sum_{n=1}^{M_N} \sum_{k=1}^{M_N} \int_0^\infty \int_0^\infty \left\{ \frac{N^2 - (n+k)N + nk}{N^2} \max\,[1, b(0)]\; b(u)\, b(v) \right.$$

$$\left. + |R_Y(u)\, R_Y(v)| \left(1 - \frac{k}{N}\right) \right\} f_{n,k}(u,v)\, du\, dv \ .$$

Now, since $\Sigma\Sigma\, f_{n,k}(u,v) \leqslant \beta(1+\beta)$, by the Lebesgue dominated convergence theorem

$$\int_0^\infty \int_0^\infty \max\,[1, b(0)]\; b(u)\, b(v) \sum_{n=1}^{M_N} \sum_{k=1}^{M_N} f_{n,k}(u,v)\, du\, dv$$

$$\leqslant \beta(1+\beta) \max\,[1, b(0)] \cdot \left(\int_0^\infty b(u)\, du \right)^2 . \qquad \text{(A-5)}$$

Next,

$$\int_0^\infty \int_0^\infty \max\,[1, b(0)]\; b(u)\, b(v) \sum_{n=1}^{M_N} \sum_{k=1}^{M_N} \frac{n+k}{N} f_{n,k}(u,v)\, du\, dv$$

$$\leqslant \beta(1+\beta) \max\,[1, b(0)]\, \frac{2M_N}{N} \left(\int_0^\infty b(u)\, du \right)^2$$

$$= O\left(\frac{M_N}{N}\right) \text{ as } N \to \infty . \qquad \text{(A-6)}$$

For the third term

$$\int_0^\infty \int_0^\infty \max\,[1, b(0)]\; b(u)\, b(v) \sum_{n=1}^{M_N} \sum_{k=1}^{M_N} \frac{nk}{N^2} f_{n,k}(u,v)\, du\, dv$$

$$\leqslant \beta(1+\beta) \max\,[1, b(0)] \left(\frac{M_N}{N}\right)^2 \left(\int_0^\infty b(u)\, du \right)^2$$

$$= O\left[\left(\frac{M_N}{N}\right)^2\right] . \qquad \text{(A-7)}$$

Finally,

$$\int_0^\infty \int_0^\infty R_Y(u)\, R_Y(v) \sum_{n=1}^{M_N} \sum_{k=1}^{M_N} \left(1 - \frac{k}{N}\right) f_{n,k}(u,v)\, du\, dv$$

$$\leqslant \beta(1+\beta) \left(\int_0^\infty R_Y(u)\, du \right)^2$$

and the lemma follows. Q.E.D.